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ABSTRACT

This pamphlet is designed to give help in developing problem solving by examples of techniques at grade levels one through six. A number of illustrations are presented of problems that are typical of the types appropriate at each grade level. For each problem, a suggestion of a solution appears with alternative methods suggested in several cases. Emphasis is given to the use of number sentences. Eight steps in problem solving are outlined for the initial development of problem-solving ability. (MP)

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PROBLEM SOLVING

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PROBLEM SOLVING

Everyone believes in problem solving. All materials that touch upon the objectives of a mathematics program emphasize the importance of problem solving. You have undoubtedly heard discussions concerning problem solving and suggestions on how to develop problem-solving ability in pupils.

This pamphlet gives help in developing problem solving by examples of techniques *at grade* levels.

What Is a Problem?

A pupil in mathematics looks upon a situation as a problem provided that he or she feels impelled toward action for some reason or other, that he or she has clearly in mind what the goal is, and that he or she is puzzled but not frustrated about what action to take. A problem can be very difficult for a pupil or it can be almost trivial. In any case, the pupil must have a desire to take action; he or she must be motivated to reach the goal. "Reaching the goal" would be called solving the problem or finding a solution to the problem. The action a pupil takes is his or her method of solution. *To have a problem is to desire to search consciously for some way to achieve a clearly understood but not immediately attainable goal.*

In this discussion of problem solving, a *number sentence* is something like the following:

$13 + 4 = 17$ or $\square + 2 = 7$ or $21 - \Delta = 14$

Number sentences may have only numerals in them, like the first illustration, or the number sentence may contain *placeholders*, such as $\square$ and Δ in the second and third illustrations. Other placeholders for numerals might be used, for example, ∇, ?, N, a, or b.

Although many use the word *problem* to mean a "story problem" or one stated in words, it is used here in a broader sense. There are many different kinds of problems for pupils; often a number sentence is just as much a problem for children as one stated in words. However, the following is called an *example:*

$$5 + 12 = 17$$

Problem Solving: How Does It Work?

What should a pupil do who is faced with a mathematical problem? *The first step is to make sure that he or she understands the problem.*

One way a teacher can make sure that a pupil understands the problem is to ask him or her to repeat in his or her own words what the problem tells. Very early in the first grade, but after the idea of combining two different sets has been introduced, the teacher can present a picture situation, or a concrete material situation, and ask a pupil to tell a number story about the situation. For example, a teacher could use a flannel board on which there was a strip with two cars parked next to it. The teacher could ask John to park three more cars; then, the teacher could ask Kathy to tell a number story about what she had seen. On the chalkboard, the teacher would have written;

_____ + _____ = _____.

By calling attention to this, the teacher would be giving a hint to Kathy that would help her to tell the story. Perhaps the teacher would want to say "Tell a number story that asks a question." Kathy or another pupil, could write the numerals in the number sentence. A later development would be for the teacher to write on the chalkboard:

_____ _____ _____.

Then, a pupil would have to decide on the operation sign as well as the numerals.

Whether a pupil wrote

$$__ + __ = __$$

or

$$__ + __ = __$$

would make no difference. Both should be considered correct, but having these two number sentences could be used by the teacher to emphasize that "2 + 3" and "3 + 2" are two "names" for 5.

At third-grade level, a problem might be as follows:

> Mrs. Johnson bought gas for her car. She paid for it with a $5 bill and got back 43 cents. How much did the gas cost?

Or, at sixth-grade level, a problem might be:

> A Little League baseball diamond is a square with a distance of 60 feet between bases. What is the least distance a boy or girl runs if he or she hits a home run?

In either case, to see if a pupil understood the problem a teacher would ask him or her to tell what the problem says in his own words. If something were left out by the pupil, the teacher could ask a leading question to prompt him or her to supply the missing information. Or, another pupil could be called on for the missing information.

The second step in problem solving is to *make sure the pupil knows what the goal is.* That is, does the pupil know what the question in the problem asks? Does the pupil know what he or she is directed to find by the problem?

Sometimes the goal is to find a missing numeral in a number sentence; sometimes it is to find a measurement. At other times, particularly in geometry, the goal is to discover a relationship, do a construction, or discover a pattern. Unless the pupil knows what the question asked in a problem means, he or she can only struggle blindly to solve the problem.

It is important to know the goal of a problem because there are so many different things to do in mathematics with the same given data. For instance, if you have two numbers, you can add them, multiply them, subtract one from the other, divide one by the other, find their ratio, or find what percent one number is of the other. The problem "chooses" one of these possibilities for the pupil. When the pupil understands this, he or she keeps an eye on the goal as he or she goes about the business of solving the problem. Keeping an eye on the goal, or keeping in mind what he or she is after in solving a problem, is one of the most important aspects of problem solving.

The third step in problem solving is *to help the pupil find a method* of attack. That is, classroom instruction should assist pupils in devising a plan for solving a problem. At this point in problem solving, a past knowledge is important. A pupil who knows very little about mathematics, who has little past knowledge, will not be able to think of a way to solve a problem. The pupil cannot find the missing numeral in a number sentence if he or she does not know addition facts; sometimes, he or she must know that subtracting is the inverse of adding, that subtraction "undoes" addition. If the pupil does not know some facts about points and lines, he or she cannot solve many problems in geometry. Past knowledge that one has learned meaningfully is easily remembered for a long time if it is put to constant use. It is this acquired knowledge that a pupil brings to bear on a problem that he or she wants to solve. Year by year, as he or she progresses in his or her studies,

a pupil has more such knowledge to use in the new problem situations he or she encounters.

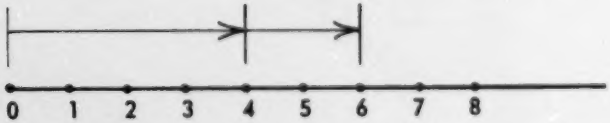
A variety of problems at all grade levels follows; for each problem a method of attack is indicated that a pupil at that particular grade level might be expected to use.

Grade 1

Early in the first grade a pupil is presented with such problem situations as

$$4 + 2 = _____.$$

If the pupil knows this addition example, there is no problem. If he or she does not have an immediate response, however, there is very little information on how he or she works his or her way out of the problem of finding the sum. Perhaps the pupil "counts on" from 4 and realizes that he or she should stop at 6. The pupil may say to himself or herself "5, 6" and then write the correct numeral in the blank. Or, the pupil may have understood the use of a number line; then he or she might start at 0 and make a move of length 4,



followed by a move of length 2. Or, he or she may reach for his or her concrete materials and combine a set of four tongue depressors, for example, with a set of two tongue depressors. Thus, a pupil has three means of attacking the problem. All are correct, although from an adult point of view the methods are at different levels of development.

Another type of example that may be a problem for a first-grade pupil is:

$$\begin{array}{r} 54 \\ +33 \\ \hline \end{array} \qquad \begin{array}{r} 98 \\ -76 \\ \hline \end{array}$$

Such examples are often presented in vertical form rather than horizontal form. The problem, here, is that a pupil should remember to begin at the *right*, adding the 1's or subtracting the 1's first; then he or she addes the 10's or subtracts the 10's. Past knowledge has to be applied. A pupil must know his or her addition facts in order to be successful in an addition example with 2-digit numerals.

Since the reading vocabulary of first graders is small, problems cannot be in printed form but must be in oral form. Pictures for which pupils tell a number story can be used; the teacher then leads pupils to the method of writing a number sentence that tells what the problem tells.

Another possibility that is often used is the dramatization of a number story. The teacher may serve as narrator, or a pupil who has been coached beforehand may serve. Such a story might be told as follows, as two pupils proceed to carry out what the narrator says:

> "Tom put four books on the table. Then Sue put five more books on the table. How many books were on the table altogether?"

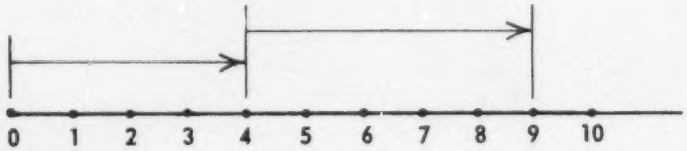
At this point, the teacher could ask, "Who can write a number sentence to tell the story?" A volunteer could write on the chalkboard

$$4 + 5 = \square$$

Then another pupil could be asked to write the unknown numeral.

Suppose a pupil says he or she got 9 in a different way and proceeds to tell that he or she counted the books on the table. A teacher might state, "That is correct. You can find the number of books on the table altogether by counting. Do you see also that you can find the number by adding?" Bring the attention of the pupil and the others in the class to adding, which is the operation that the teacher wishes the pupils to use.

The same problem could be approached by another method. The teacher could ask, "Who can show what the story tells on a number line?" Then a pupil might show something like the diagram below.



The next question would be, "What numeral is at the end of the second move?"

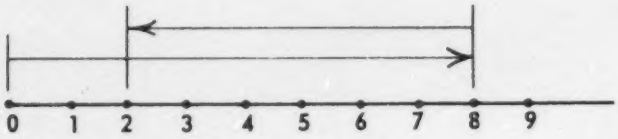
Another stage in developing skill with the foregoing type of problem would be for the teacher to tell a number story. For example:

> Mother had eight boxes of cookies. She gave six boxes to the children. How many boxes did she have left?

After making sure that the pupils understand the problem (by having one retell the story) and know the goal (by asking another to tell what the question in the problem is), the teacher can call on a volunteer to show how to get the answer. One pupil might write a number sentence as follows:

$$8 - 6 = \Delta$$

Another pupil might use a number line:



A different problem for a teacher to tell is:

> Joe is saving boxtops. He has 6, but needs 10 to get a baseball bat. How many more boxtops must he save?

This is a difficult problem at the first-grade level. A teacher should make sure that pupils both understand what the problem is about and are able to tell in their own words what the problem asks them to find.

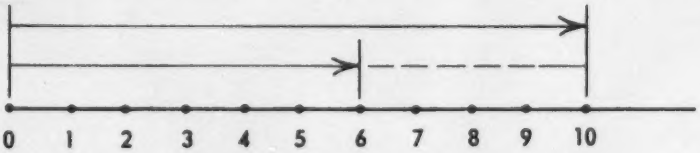
A number sentence for the problem is:

$$10 - 6 = \square$$

Pupils can find the unknown numeral in two ways. They can think "What number added to 6 gives 10?" Or, they can think "10 less 6 is what number?" Some pupils might make a mental jump and write the number sentence:

$$6 + \square = 10$$

Of course, a teacher should *accept* this as a correct number sentence leading to a correct solution. A number line for this problem might be the following:



Notice that no arrowhead appears on the dotted line. This allows for both an additive and a subtractive way of arriving at the correct answer.

Grade 2

Let us emphasize again that a problem is an individual sort of situation. In this pamphlet, problem is used in a broader sense than the usual interpretation of a situation described in words. Early in the second grade, work begins with teaching *patterns* in the sequence of whole numbers. Pupils should understand which number comes before a given number and which number follows immediately after a given number. Exercises might be given such as the following, where pupils are asked to write the correct numerals in the blanks:

____, 8, 9, _____;

_____, 7, _____, _____, _____;

_____, 16, _____, _____.

The first example has two whole numbers to serve as a hint, whereas the second and third examples have just one. The latter are more difficult because more thinking is involved.

As the second-grade year progresses, more difficult exercises of the pattern sort can be given:

_____, 47, _____, _____, _____;

_____, 89, _____, _____, _____;

157, 158, _____, _____, _____.

These exercises require a pupil to go from one set of 10's to another in counting. He or she has to "turn the corner," so to speak, from the 80's, for example, to the 90's. After odd numbers and even numbers have been taught, there are other sequences with blanks to fill that can be used:

1, 3, 5, _____, _____, 11, _____, _____;

4, 6, 8, _____, _____, _____;

3, 13, 23, 33, _____, _____.

The same sequences could be presented by means of a number line. Put on the chalkboard the following diagram:

Leave as many gaps as you think wise for your class. Ask a pupil to write the numerals missing between 2 and 5; ask another to write the numeral missing between 5 and 7. Proceed in this way until you have the number line completed as far as you like. There are many variations of this exercise that a teacher can easily devise.

An interesting variation of a standard problem develops into a search for patterns.

> Jane bought a pen for 15 cents and a pencil for 3 cents. How much did she spend?

This is a straightforward one-step problem (all that is involved is one operation); a pupil might write the number sentence

$$15 + 3 = ?$$

and say Jane spent 18 cents. The problem is solved, but there is little challenge in solving such a problem. A different situation may be added to this problem. Perhaps the day of the purchase was a sale day. The conditions of the sale are in the following sign.

SALE	
15¢ Pens	*Now* 12¢
3¢ Pencils	*Now* 2¢
Limit of four of each to a customer	

Under the conditions of the sale, what is the cost of the largest possible purchase Jane could make? How many pens could she buy? How many pencils? What is the cost of four pens? What is the cost of four pencils? What is the total cost of four pens and four pencils?

More possible questions are: How many combinations of pens and pencils could Jane buy? What is the cost of each combination? Here is a new problem. You could list the combinations to see whether there were any patterns. For example:

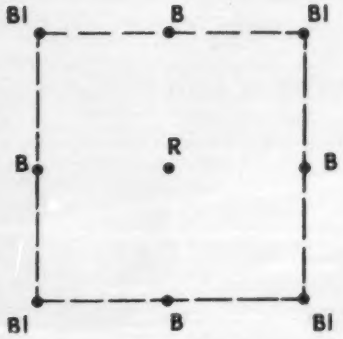
4 pens and 0 pencils
4 pens and 1 pencil
4 pens and 2 pencils
4 pens and 3 pencils
4 pens and 4 pencils

This pattern runs out; start another pattern. There are 24 possible purchases.

More possible questions, using the same sale data, are: Jane spent 26¢. What combinations of pens and pencils might she have bought? Or, Barbara spent 20¢. What do you think she bought?

In short, a bit of imagination used with rather simple data of the sale of pens and pencils can lead to interesting investigations by pupils and work in one of the solving activities considered important today.

Geometry in the second grade explores relationships of points, line segments, and common geometric figures (square, triangle, circle). The main objectives of the instruction are to lay a foundation for geometric principles and to teach for an understanding of the characteristics of common geometric figures. A typical exercise would present a diagram like the one on the following page; attention of a pupil could be directed to points by referring to a "blue dot" or a "red dot."



B means "blue dot"

Bl means "black dot"

R means "red dot"

Following are typical directions and questions. Trace the dotted line segments joining black dots. What can you say about the lengths of these line segments? What is the name of the shape shown?

Are the points shown by blue dots on the square? Mark with your pencil as many line segments as you can joining pairs of blue dots. How many such line segments are there? Is the red dot on one of the line segments? Is it on two of the line segments? Is it on three?

For the following questions, it would be well if a teacher had placed the diagram up to the point of the last paragraph on the chalkboard; pupils then could be called upon to point out their answers to the following questions.

How many squares are shown in the diagram? (6) Barbara, would you point out all the squares you have found? How many squares have the red dot at one corner? Jim, would you show us these squares? How many triangles are shown in the diagram? (12) Bobby, would you show us all the triangles you found? How many triangles have the red dot at the corner? (4) If the red dot is at a corner, is it on a side of a triangle? Is it on two sides of a triangle? How many triangles have the red dot on a side? (8).

There are more questions that could be asked about the same diagram and more line segments that could be represented in it. The sample given in the preceding paragraphs is sufficient to illustrate the purposes of such a problem-solving situation for second-grade pupils. There are several ideas involved. Pupils should recognize the common geometric shapes and become acquainted with some of their characteristics, such as, a square and a triangle are sets of points; a square has four sides of equal length; a triangle has three sides. Pupils should be involved in discovering relationships between points and lines; in the case of the diagram under discussion, it would be points and line segments. Pupils should become sensitive to extending their ideas of various relationships. For example, if a point is at a corner (vertex) of a triangle, it is on two sides of the triangle.

Notice that the foregoing type of geometric exercise does not lead to any number sentence as a method of attack. The method of attack is thinking about or visualizing possible relationships. Counting is involved much more than any operation such as addition or subtraction. In a way, geometry requires more insight and a firmer grasp of basic principles than, say, arithmetic.

Since the reading vocabulary of second-grade pupils grows during the school year, toward the end of the year a teacher might present a problem like the following:

> Susan and Bob counted 15 sheep, 6 pigs, and 3 cows on the farm. How many fewer pigs than sheep were there?

Notice that the problem contains extra data; the three cows have nothing to do with the question asked in the problem. Such information has the label "extraneous data." Definite teaching and experience for developing ability to handle a situation of this

sort are needed. The first time pupils are asked to solve a problem of the above type, a teacher should ask questions similar to the following. What animals are mentioned in the question asked in the problem? Are cows mentioned in the question? Do we need to know how many cows there are to answer the question asked? So, the number of cows on the farm is extra information. What animals do we need to know about to answer the question?

The method of attack is to write a number sentence. A teacher might put on the chalkboard a skeleton form,

____ ____ = ____,

and ask different pupils to fill in the blanks. A teacher might say, "Tell in your own words what the problem asks us to find." Then, "Do you think we will use addition or subtraction to find the answer?" Have a pupil put the correct sign of operation between the first two blanks. A pupil can be asked to write the numerals in the remaining blanks.

Grade 3

During the third grade, the work with place value, algorithms—for addition, subtraction, multiplication, and division over the set of whole numbers—and "finding the missing numeral" sentences continue. Each of these phases of the year's study presents problems of various levels of difficulty to individual pupils, but in this section let us consider primarily problems according to the usual interpretation of the word.

It is at this grade level that most programs introduce the two-step problem, a problem for which the solution requires two operations. One of the ways to lay a foundation for solving such problems is to begin with a sequence of two problems, so that a pupil

has to use the information obtained from the first problem in the solution of the second problem. For example, consider the following two problems.

> *Problem 1.* There are 14 boys and 15 girls in Mrs. Brown's third-grade class. How many pupils are there altogether?
>
> *Problem 2.* One day, 6 pupils were absent. How many pupils were present?

A number sentence for Problem 1 is

$$14 + 15 = \Delta,$$

and 29 pupils answers the question asked in the problem. When Problem 2 is attacked, a pupil must find, or remember, the total number of pupils in the class before writing a number sentence that tells about the problem:

$$29 - 6 = \square.$$

Many pupils need help at this point. Some suggested questions to ask are: What number must you know before you can find how many pupils were present? Does one of the other problems tell you this? Or, if the second question elicits no satisfactory response, did you find how many pupils are in Mrs. Brown's room in the last problem? Can you use this number in Problem 2? It may seem that the last two questions practically tell a pupil the answer, but this is not true so far as a pupil struggling to understand is concerned.

Much experience with the preceding sort of problem solving serves as a basis for attacking a problem such as the following:

> There are 29 children in our class. On Monday three boys and two girls were absent. How many children were present Monday?

Possible questions to ask in working with a class are: What does the problem ask us to find? Do we know how many pupils were absent on Monday? Can we find out? What is a number sentence that will tell us? It is

$$3 + 2 = ?$$

Have we answered the question asked in the problem? How can we find how many pupils were present? Who can write a number sentence that tells?

There are two number sentences that might be written. Since the numbers used in the problem are so small, a natural number sentence for most pupils to write is

$$29 - 5 = n,$$

which will yield the answer to the question asked. Another possible number sentence is

$$29 - n = 5,$$

one that shows more mathematical insight than the other. A pupil who writes such a second number sentence probably has more feeling for mathematics than his or her classmates (and should be given some individual attention so far as enrichment materials are concerned).

Some ingenuity must be used to present problems for pupils to solve in which the missing numeral, represented by a frame or a letter, appears in different places in a number sentence. The following problems are illustrations.

> One day Bob sold 24 roses to customers at the flower shop. When he went home, there were 84 roses left in the shop. How many roses were there before he sold some of them?

In analyzing such a problem, a pupil has several decisions to make. What will the size of the answer be relative to the other numbers given? What operation should be used? How can the problem be summarized by a number sentence? Clearly, one number sentence is

$$? - 24 = 84,$$

and this is the one most teachers would like to see a pupil write. A pupil who carries his mental analysis further might write

$$? = 84 + 24,$$

which is also correct.

Here is another problem:

> Alice says 108 chairs will be needed in the room when the show is given. There are 32 chairs now. How many more chairs will be needed?

A number sentence for the problem is

$$32 + ? = 108,$$

but a pupil might take a larger step in analysis and write

$$? = 108 - 32.$$

The latter number sentence should be considered correct also. In short, a teacher must be willing to accept correct ways to solve a problem no matter how different they are from preconceived plans or ideas for problem-solving experience.

A great deal of emphasis is placed on estimating answers in the elementary school program. One reason for such emphasis is that pupils can be taught to test

whether an answer is reasonable. For example, given the following problem,

> One airplane carries 189 passengers, and another carries 95 passengers. How many fewer passengers does the second plane carry?

there are several questions that can be asked to encourage estimation. Final questions might be "What number sentence can you write that tells about the problem? 189 - 95 = ? Then your answer will be about what number? Find the answer. Was your estimate about right?

A problem that would be quite challenging for third-graders is the following:

> Mr. Jones has a board 20 feet long. What combination of 5-foot, 4-foot, and 2-foot pieces could Mr. Jones cut without wasting any part of the board? Copy the table below, and write numberals to show the number of pieces he could make from one board.

5-foot pieces	0									
4-foot pieces	1									
2-foot pieces	8									

There are 10 possible combination of pieces. The problem can be left unstructured by presenting just the table, or a teacher may fill in a few of the entries. A problem of this sort needs preliminary experiences with simpler problems having partially structured solutions, but the above problem is typical of the unusual sort of situation a new mathematics program is likely to contain.

Grade 4

Fourth-grade problems involve not only more mathematics but also more vocabulary, for both the technical and reading knowledge of pupils has increased. The idea of an average is introduced and illustrates a two-step problem that involves both addition and division. A typical such problem for this grade is:

> Kathy's scores in a dart game were 45, 37, 34, and 52. What was her average score?

Unless a pupil brings the past knowledge of what an average is to an understanding of what the problem tells him or her, there is no way in which he or she can solve it. A pupil has to know that the average is the sum of all the scores, divided by the number of scores. Once he or she realizes this, and once he or she notes that the key word in the statement is "average," it is possible to write two number sentences:

$$45 + 37 + 34 + 52 = ?,$$

$$\text{Sum} \div 4 = \text{Average}$$

Finding the missing numeral in the first sentence and using it in the second sentence, the pupil is able to find the desired average.

Another two-step problem that requires addition and subtraction is the following:

> Sarah was given $5 for her birthday. She bought her dog a new leash for $1.50, a tie-out chain for $1, and a tie-out stake for 50¢. How much money did Sarah have left?

In this problem the analysis is more difficult than it is to get the answer. Most fourth-grade pupils will

be able to understand the problem, as shown by their ability to tell it in their own words, but many will fail to keep an eye on what the question in the problem asks. Without doing so, the way to the goal—a solution—is lost. Clearly, two number sentences can be written for the problem:

$$\$1.50 + \$1 + 50¢ = \$3,$$

$$\$5 - \$3 = \$2;$$

the exact amount left is 2 dollars.

Research shows that the most difficult two-step problem is one involving multiplication and division. Such a problem is the following:

> Barry wants some large prints made of nine pictures. He can have three large prints made for $1.20. How much will the large prints cost?

To solve this problem, a pupil concentrates on the question asked. The pupil thinks: How can I find the cost of the large prints? What do I need to know? (The cost of one print and how many prints; or, what multiple 9 is of 3.) The amount of thinking involved seems to be of a higher level than that required for an addition-subtraction problem for instance. A pupil might write the number sentences

$$\$1.20 \div 3 = ?,$$

$$9 \times ? = n,$$

which would lead the pupil to a correct solution, or he or she might write

$$9 \div 3 = ?,$$

$$? \times \$1.20 = _____,$$

which also would lead to the correct result. It seems that the method of attack used by a pupil on this problem depends more upon how his or her teacher has demonstrated solution of problems and developed problem-solving ability than upon the mathematics. In any case, the point is emphasized that there are many paths to a correct answer and many correct methods of attack.

A type of problem that is appearing in elementary school materials and that is quite different from any presented so far is the following:

> Harry invited this set of boys to his birthday party: Tom, Pete, Fred, Bill, Bob, and Dick. He also invited this set of girls to his party: Susan, Nancy, Joan, Mary, Alice, and Linda.
>
> Fred, Dick, Susan, and Joan arrived late for the party. Dick, Joan, Linda, and Nancy left the party early. How many boys and girls arrived late *and* left early. How many boys and girls arrived late *or* left early?

In understanding the problem, a point to bring out is that the set of boys and girls who arrived late might have some members in common with the set of boys and girls who left the party early. Not only does the problem contain extra information—extraneous data—it is a problem that does not yield to a number sentence method of attack or to computations. Essentially, this is a reading and counting problem. The information contained in the first paragraph of the statement of the problem has nothing to do with answering the questions asked, that is, solving the problem. All the relevant information is contained in the second paragraph.

The two questions contain two key words, which are in italics. The first question asks for the

number of boys and girls who both came late and left early. One looks for the names that appear in both sets. These are Dick and Joan. Thus, the set of boys and girls who came late and left early consists of Dick and Joan. Some will recognize in this question the intersection of the two sets. The number who came late and left early, therefore, is two.

The key word in the second question of the problem is *or*. Consider what *or* means. It means the boys and girls who came late or who left early or who both came late and left early. In short, the set of boys and girls mentioned in the last paragraph of the problem. This set consists of Fred, Dick, Susan, Joan, Linda, and Nancy. Many will recognize this set as the union of the two sets of the second paragraph of the problem. The number of boys and girls who came late or left last is six.

Problems such as these, among others, serve to lay a foundation for the operations with sets before symbolism is introduced. Much experience with these ideas underlying set symbolism is necessary before the symbols are used. Although an adult may find problems such as the foregoing very simple, children may not find them so. There is a high level of mature reasoning involved. Furthermore, such problems can become quite complex with only a small extension of the given data. For example, such problems might involve three sets instead of two sets.

Grade 5

The same types of problems that have been considered for earlier grades are continued in the *fifth grade*. In addition, there are problems that apply the new mathematical concepts introduced in the fifth grade. For instance, the study of ratio is more thorough at this grade level than in the third or fourth grades. Toward the end of the year, a pupil

might be expected to solve a problem such as the following:

> The tiles in a hallway of the school are to be replaced during the summer vacation. The pattern for the new floor calls for one red tile for each five white tiles. If, 1,000 white tiles are needed, how many red tiles will be needed?

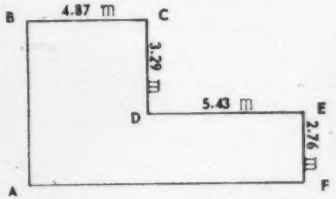
There are two key phrases, "one red tile for each five white tiles" and "1,000 white tiles are needed." The first tells a pupils that the ratio of red tiles to white tiles is 1 to 5. Thus, he or she knows that the number of red tiles is 1/5 the number of white tiles; the pupil can write the number sentence

$$1/5 \times 1{,}000 = ?$$

which leads to the answer to the question, 200 red tiles.

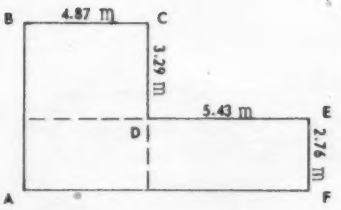
A geometry and measurement problem is the following:

The polygon at the right has right angles. What is the length of side AB and the length of side AF?



A teacher never knows what a pupil thinks who solves a problem of this sort. What does the pupil remember? What principles does he or she apply? What does he or she do intuitively, almost without reason. One does not know the answers to these questions. Essentially, the problem tells a pupil certain measurements and that right angles are present. Does a pupil realize the right angles guarantee the figure contains rectangles? Does he or she recall that the opposite sides of a rectangle have equal lengths? Pupils do realize, however, AF = BC + DE, and AB = CD + EF.

Some pupils may copy the diagram and actually put in the dotted lines to indicate the rectangles; others may do this mentally. However, properties of rectangles are the key to a solution of the problem.



This problem can be turned into a difficult two-step problem by replacing the question stated with "What is the perimeter of the polygon?" Such a question requires a pupil to make use of even more past knowledge than the stated problem specifies.

An example of a problem that requires interpretation of what the remainder means in a division is:

> Small buses are used to carry children to a picnic. Each bus can safely carry 41 children. There are 570 children going to the picnic. How many full buses will be needed? How many children will there be for a partially filled bus?

A correct number sentence for the problem is

$$570 = (? \times 41) + R,$$

where R represents the remainder. There will be 13 full buses, and 37 children in the partially filled bus. Altogether, 14 buses will be needed.

A problem with extraneous data is the following:

> The water level in a tank rose from the 3.38m mark to the 5.25m mark on a measuring stick. Later the water fell from the 5.25m mark to the 2.88m mark. What is the difference between the highest and lowest level of the water?

Not only is the problem difficult to read, with the mixed decimals and physical-situation terms, but also there is difficulty in deciding what *are* the essential data. A number sentence that could be expected is:

$$5.25 - ? = 2.88$$

A teacher might ask what other number sentence might have been written. Any one of the following would be correct:

$5.25 - 2.88 = ?$; $5.25 = ? + 2.88$; or $5.25 = 2.88 + ?$

Naturally, problems in making graphs of a variety of sorts, interpreting graphs, and work with averages would be extensive. A problem on averages that is typical at fifth-grade level is:

> Chemicals are used a great deal in the fighting of forest fires. If an average of 3,000 liters was used for each of 322 fires, what was the total number of liters of chemicals used?

A number sentence for the problem is

$$3{,}000 \times 322 = N,$$

but involved in finding such a number sentence is understanding of what an average is and what it means. In fact, a pupil must understand how to recover the total number from the average and the number of items.

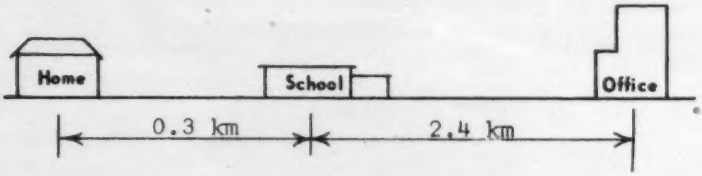
Grade 6

Rather early in the sixth grade, problems with two steps and using decimal fractions can be introduced, such as:

> It is 0.3 km from Mrs. Brown's home to the school where her son is a pupil. From the

school to Mrs. Brown's office it is 2.4 km farther along the same road. What is the length of a round trip between Mrs. Brown's home and her office?

Sometimes a drawing that tells about the problem will help children to understand it. This can be a rough drawing, such as the one below.



Be sure pupils understand what a round trip means, for this understanding is the key to answering the question asked in the problem. Some pupils, particularly if a drawing is used, will write the following number sentence for the problem,

$$0.3 + 2.4 + 2.4 + 0.3 = \square;$$

others will think of finding the length of one way and then doubling it:

$$0.3 + 2.4 = \square,$$

$$2 \times \square = \Delta$$

Different methods of attack may be correct, and it is usually worthwhile to use the necessary class time to bring these out.

Problems involving different kinds of numbers can be used in the sixth grade. The problem that follows is an example; in addition, it is an illustration of the extraneous data type of problem.

> One day Mr. Edwards drove 270 km; the next day he drove 480 km. One the second day, he spent 5 2/3 hours driving. To the nearest km per hour, what was his average speed that day?

This problem involves many decisions for a pupil. The pupil must decide that the 270 km is not needed to answer the question asked in the problem. He or she must know what an average speed is. He or she must know also that average speed is found by dividing the total number of km traveled by the number of hours driving. He or she must know what rounding is involved in finding the average speed "to the nearest km per hour." When a problem is analyzed this way, one wonders how a pupil ever solves it! And one can understand better why some pupils have difficulty with problems. Once the decisions are made, it seems as if a number sentence for the problem would be easy to write:

$$480 \div 5\ 2/3 = N$$

A type of problem that has appeared in elementary school materials over the past few years is one with incomplete data. Such a problem, in a way, is the opposite of a problem with too many (extraneous) data. A pupil supplies reasonable data and proceeds to find the answer to the question asked in the problem. An illustration of this sort of problem is:

> Find the number of square centimeters in a pane of glass that is 2.5 meters wide.

Of course, the datum needed is the length of the pane of glass.

Currently percents are considered special ratios, it is quite natural to have such problems appear in the sixth grade. The following problem might be considered typical:

At the end of 1 year a compact car that originally cost $3,424 has a value of $2,568. What is the percent of decrease?

Number sentences can be used to "outline" the solution of the problem:

Amount of decrease: $3,424 - $2,568 = N

Percent of decrease: N is what percent of $3,424?

or $\frac{N}{3,424} = \frac{?}{100}$

The method of actually finding the percent would depend upon what method a particular teacher uses with his or her class.

Summary

In this section devoted to methods of attack, a number of illustrations have been presented, at each grade level, of problems that are typical of the types appropriate to that grade level. In each case, a suggestion of a solution appeared, along with alternative methods in several cases. Much emphasis has been given in this section to the use of number sentences, in harmony with current practices. What has been omitted from the illustrations has been much of the work with measurement, graphs, and geometry. These topics do not lend themselves to short, clear, or concise examples. The material is something that has to be developed over several class lessons, rather than during a part of one lesson.

The illustrations intend to show these points. *Strip a problem to its essentials.* Almost every problem appears with a certain amount of window dressing that puts the problem in context and that tells to what the mathematics is being applied, but this has little to do with the method of attack.

A pupil soon learns to disregard extraneous data and to concentrate on the relevant facts presented in the problem.

Recall past knowledge. Solve a problem in mathematics by bringing to bear past successful methods of attack, by recalling knowledge about numbers and operations with numbers, by sometimes making a small extension of these past methods, or by concentrating on key words and their mathematical meaning.

Write a number sentence that tells about the problem. A number sentence summarizes what the problem tells and what the problem asks.

However, a number sentence is not always practicable. In such a case, *search for a pattern.* It may be that, by examining the data, certain regularities become apparent; then the regularity can be generalized and tested. Or, the pattern discovered may suggest a way of solving the problem.

In the case of geometry problems, *look for a relationship.* The search for a relationship is closely associated with the search for a pattern. A relationship observed is generalized and the generalization tested in all possible situations.

Sometimes none of the foregoing methods serves to solve the problem. Then, *try a special case to see if it suggests a method of attack.* For example, if the problem is about rectangles, try to solve it in the case of squares. If the problem is about parallelograms, try to solve it in the case of rectangles. If the problem is about fractional numbers, try to solve it using whole numbers.

The method of attack, the plan, serves as a general outline for a problem, but it does not fill in the details. Perhaps something was overlooked. Perhaps an error was made. Carrying through the details of solving a number sentence, for example, a

pupil does the necessary arithmetic. He or she can check his or her work at this point by comparing the result with his or her estimate. If there is a wide divergence between his or her estimate and his or her result, he or she has a signal that something is wrong. The pupil should go back over his or her work to try to find where he or she went astray.

Finally, *look back at the solution.* As has been mentioned earlier in the discussion of problem solving, looking back at the solution of a problem is often skipped. An important part of instruction in problem solving thus is missed. A pupil who reexamines his or her solution may find a shorter, neater way to solve the problem. In such a way the pupil consolidates or reorganizes his or her knowledge of mathematics. He or she adds to his or her store of knowledge that can be called upon when faced in the future with the necessity for solving a problem. Sometimes a look back suggests an alternative method of solution. Sometimes the examination of a solution suggests a generalization, or another problem to solve.

Another phase of looking back is that pupils and teacher together can bring out the relationships between the problem just solved and past mathematics studied. It is a rare problem that exists in isolation, that is not related in some way to what pupils have studied earlier. Such a reexamination of a solution underscores for the class that all mathematics is related. Pupils grow in the realization that mathematical knowledge is cumulative; past learnings are as useful in this grade as they were when they were first studied.

Concluding Statement

Steps in problem solving can be summarized as follows:

Strip a problem to its essentials.

Recall past knowledge.
Write a number sentence that tells about the problem.
Search for a pattern.
Look for a relationship.
Try a special case in seeking a method of attack.
Carry out a plan.
Look back at the solution.

If this were all there were to problem solving, teaching the preceding steps, there would be no need for all the discussion of problem solving in teaching mathematics. Every experienced teacher knows there is much more to developing problem-solving ability than knowing these steps. Also, every experienced teacher will see the shortcomings in these steps.

It is true that every time a person suggests steps for successful problem solving, someone else devises a problem to which the steps do no apply. Nevertheless, teachers should be aware of the steps illustrated herein; they have proved to be very successful in mathematics. However, a teacher should not conclude that these steps include all there is to know about problem solving.

There is a final suggestion to make: It is that instruction allow for different pupil solutions. A modern mathematics program should develop a flexible approach to problem solving in pupils. A desirable outcome is to have a pupil approach a problem with his or her past accumulated knowledge and no preset tried-and-sure way of solving the problem. A desirable frame of mind in a pupil is that where he or she is ready to innovate and experiment (shift his or her attention from one point to another) to find a successful method of attacking the problem.

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